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COMPUTATION OF APPROXIMATELY OPTIMAL CONTROL

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ABSTRACT

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This research is concerned with the problem of optimal control of dynamic systems, where optimality here implies the minimization of some specified cost integral during the control process. In this study a practical computational procedure is developed for obtaining feedback control laws which are approximately optimal.

The maximum principle of Pontryagin provides the theoretical basis for the proposed synthesis procedure. Application of the maximum principle in a given control problem yields an optimal control law which is a function of both the known system state vector and an associated unknown adjoint vector. At each point in state space this adjoint vector can be identified as the negative gradient of a certain scalar function of state, the optimal cost function, defined to be the minimal cost obtainable when that point serves as the initial condition for the dynamic process. Hence, knowledge of this optimal cost as a function of state would be sufficient to enable realization of optimal control.

The basic concept of the synthesis procedure described here is the functional approximation of the optimal cost and the use of this approximation in place of an exact representation to obtain a control law which is nearly optimal. This approximation is obtained by computing the optimal cost at a number of points along each of several individual optimal trajectories, tabulating these data, and then fitting a polynomial in the state variables to the stored data points by the method of least-squares. A near-optimal control synthesis is then achieved by implementing the control law obtained by application of the maximum principle, using an approximation of the adjoint vector obtained from the gradient of the polynomial representation of the optimal cost function.

The computer time and memory requirements of this synthesis procedure are such that application of the method to systems of order up to four or five is feasible with current computers. The resulting control law is generally a simple function of the state variables and therefore readily implemented by an on-line controller of modest computing capacity. In addition, modification of the control law to insure stability is easily accomplished.

Three computational examples illustrate the procedure and verify its applicability.

Author

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LIST OF SYMBOLS

$a_{ij\ldots k}$	coefficient of approximating polynomial
$\underline{c}$	feedback control law, a function of $\underline{x}$, of dimension $r \times 1$
$\underline{c}^*$	optimal feedback control law, a function of $\underline{x}$
$\underline{\tilde{c}}$	optimal control law, a function of $\underline{x}$ and $\underline{\lambda}$
d	distance measure
$\underline{d}$	system distribution vector of dimension $n \times 1$
$\underline{f}$	system function vector of dimension $n \times 1$
F	system transition matrix of dimension $n \times n$
g	function defining switching surface
H	Hamiltonian function
$\hat{H}$	pseudo-Hamiltonian function
I	optimal cost function
$\hat{I}$	approximation to optimal cost function
J	cost function
ℓ	loss function
L	dimensionality of least-squares matrix
m	degree of approximating polynomial
M	region of state space
n	dimensionality of system
N	number of data points
p_i	polynomial of degree i
Q	constant matrix of dimension $n \times n$
r	dimensionality of control vector
S	least-squares matrix of dimension $L \times L$

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I. INTRODUCTION

A. OUTLINE OF THE PROBLEM

As an illustration of the type of control problem to be treated in this investigation, consider the following example. In order to successfully carry out an astronomical observation, an artificial satellite is required to maintain a fixed attitude in inertial space. However, the satellite is subjected to disturbance torques from such varied sources as micrometeor impacts, the motion of internal mechanical parts, and electromagnetic coupling with the earth's field. To counteract these torques the satellite is equipped with reaction wheels and gas jets for attitude control. After any momentary displacement caused by such disturbances the control system must return the satellite to the prescribed attitude. Further, it is required that each maneuver be performed in some optimal manner; for example, it may be required that a weighted sum of the integral squared attitude error and the amount of fuel consumed in the maneuver be minimized. The control signals to the actuators are to be generated by a small on-board digital computer, or controller. The question to be answered is this: What control law can be mechanized in the controller to provide optimal attitude control of the satellite?

Generalizing from this specific example, consider a dynamic system described by the system of first-order ordinary differential equations

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) ,$$

where $\underline{x}$ is an n-dimensional vector of state variables and $\underline{u}$ is an r-dimensional control vector. The control $\underline{u}$ is to be generated as a function of the state variables by an on-line controller according to some feedback control law $\underline{u} = \underline{c}(\underline{x})$ (Fig. 1). What is sought is that particular control law which insures that the system will fulfill its prescribed mission from any arbitrary initial condition in a given limit the computing capability of the controller, there are often definite advantages to be gained by reducing its size and complexity. Further, since the control signals are to be generated continuously as functions of the current state, the controller, if it is a digital device, must repetitively compute the control at intervals of very short duration in comparison with the dynamic response time of the system being controlled. Hence, for practical synthesis the sought-after control law should be of a form which requires neither extensive memory capacity nor long computational time for its realization by the on-line controller. These considerations are entirely qualitative, of course, and specific controller design constraints must originate with the particular application.

The need for a second modification to the original problem is occasioned by the practical reality that even though it may be theoretically possible to obtain the optimal control law with great accuracy, it is almost never feasible to do so. In the first place, the computational effort required invariably increases rapidly with the degree of accuracy sought, and the point is soon reached when the quest for further accuracy becomes economically unjustifiable. Secondly, the more nearly exact the representation of the optimal control law becomes, the more difficult is its realization by the controller, in general, since the requirement for simple on-line computability discussed in the preceding paragraph cannot be met. Finally, for many systems, the sensitivity of the performance to changes in control is small in the vicinity of the optimal point, so that even a control law which rather crudely approximates the true optimal law can often yield near-optimal results. For these reasons, then, a computational procedure which yields merely an approximate synthesis of the optimal control will be quite satisfactory, provided only that the quality of the approximation is sufficiently good to effect a control which results in fulfillment of all mission requirements. Of course, the degree of approximation required for satisfactory control depends on the particular problem under consideration - the system, the cost function, the mission and required performance specifications. Hence, a practical general synthesis procedure is very slow and the on-line controller very large, and hence is applicable to only a small class of problems. Another extension of the methods for solving the optimal trajectory problem involves the generation of several such solutions for initial points spaced around the boundary of the region of desired control, with some form of interpolation between these known optimal trajectories yielding the control as a function of the current state. This procedure, sometimes known as "flooding", is described in some detail by Kipiniak [Ref. 12]. Finally, there is Bellman's dynamic programming [Refs. 13, 14] which in essence comprises an efficient search process among all possibly optimal controls to find those which are, in fact, truly optimal.

C. NEW RESULTS

In these pages is described a new computational procedure for obtaining feedback control laws which are approximately optimal. This method is based primarily on the theoretical results of Pontryagin [Ref. 1]. Application of Pontryagin's maximum principle in a given control problem yields an optimal synthesis in the form of a control law which is a function of both the known state vector and the unknown solution vector to the so-called adjoint system of differential equations. This adjoint vector, considered as a function of state, can also be identified as the negative gradient of a certain scalar function, the optimal cost function, defined at each point $\underline{x}$ to be the cost incurred during the course of the dynamic process when starting in state $\underline{x}$ and using optimal control. If this optimal cost function were a known function of state, therefore, a representation of the adjoint vector could be obtained by partial differentiation and the optimal control law thereby written as a function of the state variables alone, yielding a feedback synthesis of the desired form.

Unfortunately, the optimal cost function can be exactly determined by analytic methods for only a very few special classes of problems. It is possible, however, to compute a functional approximation of the optimal cost and to use this approximation in place of an exact representation to obtain a control law which is approximately optimal.

a type of stability for this control problem is defined and it is shown how the control law may be modified to assure stability of the controlled system. The computational feasibility of the proposed synthesis procedure is discussed in Chapter VII, and comparisons with other general methods for solution of the optimal control problem are drawn. Extensions of the proposed procedure are also indicated. Finally, Chapter VIII contains some computational examples which illustrate and verify the synthesis procedure.

Let X be a bounded region of state space, the n-dimensional vector space of $\underline{x}$, within which a solution to the control problem is desired. Let T be a smooth q-dimensional manifold defined in X, where $q \leq n-1$. The time of termination of the dynamic process defined by (2.1), t_f, is to be determined as that time at which the state point $\underline{x}$ first touches T, and hence T is called the terminal manifold. If $q = n-1$, T is an (n-1)-dimensional hypersurface in X; if $q = 1$, T is a curve in X; if $q = 0$, T is a point in X.

The class of admissible control functions $\underline{u}(t)$, $t_o \leq t \leq t_f$, is taken to be the class of arbitrary piecewise continuous functions ranging in a closed bounded set U of the r-dimensional space of the control variables.

Let $\ell(\underline{x},\underline{u})$ be a non-negative loss function representing the rate of accumulation of cost, or penalty, during the course of the dynamic process. Thus ℓ is the time derivative of the quantity to be minimized by the proper choice of control. As with the system function $\underline{f}$ discussed above, the possibility that ℓ is an explicit function of time is not excluded by this representation.

The vector function $\underline{f}(\underline{x},\underline{u})$ and the function $\ell(\underline{x},\underline{u})$ are assumed to be defined and continuous on $X \times U$, and continuously differentiable with respect to $x_1,\ldots,x_n$.

B. STATEMENT OF THE PROBLEM

The optimal control problem may now be formulated. Given any initial point $\underline{x}(t_o) = \underline{x}_o$ and a terminal manifold T in X, among all admissible controls $\underline{u} = \underline{u}(t)$ which transfer the state point $\underline{x}$ from $\underline{x}_o$ to T, find one for which the cost integral

$$J(\underline{x}_o, [\underline{u}(t)]) = \int_{t_o}^{t_f} \ell(\underline{x}(t), \underline{u}(t))dt \qquad (2.4)$$

takes on the least possible value. (The brackets around $\underline{u}(t)$ indicate that J is a function of the entire time function $\underline{u}(t)$, $t_o \leq t \leq t_f$; i.e., J is a functional.) Here $\underline{x}(t)$ is the unique solution to (2.1)

A solution to the problem as stated consists of finding, for each point $\underline{x}_o$ in X, an associated control function $\underline{u}(t)$, such that the resulting solution curve, or trajectory, of the system (2.1) is optimal in the sense of minimizing the cost integral (2.4). A well known and easily demonstrated property of optimal trajectories, however, is that every portion of an optimal trajectory is itself optimal, a fact often referred to as the "principle of optimality" [Ref. 13]. Therefore, if the state point is at $\underline{x}_o$ at any instant of time, no matter along what trajectory it arrived there, the subsequent motion must be along the optimal trajectory emanating from $\underline{x}_o$. Hence, the value of the optimal control $\underline{u}$ at the instant the state point passes through $\underline{x}_o$ depends on $\underline{x}_o$ alone. Therefore, rather than find the totality of control functions $\underline{u}(t)$ corresponding to all possible choices of $\underline{x}_o$ in X, it suffices to find the single optimal control function $\underline{u} = \underline{c}^*(\underline{x})$, where $\underline{x}$ ranges over X. This result formulates the solution to the optimal control problem as a feedback control scheme. To the extent that the function $\underline{u} = \underline{c}^*(\underline{x})$ can be synthesized, optimal control of the system can be realized.

A final assumption is that the system can be observed, in the sense that at each instant t, $t_o \leq t \leq t_f$, the state vector $\underline{x}$ can be either measured directly or computed unambiguously from measurements of related system quantities.

adjoint system of differential equations:

$$\dot{\underline{\lambda}} = - \underline{f},_{\underline{x}}^{T} \ (\underline{x},\underline{u}) \ \underline{\lambda} + \ell,_{\underline{x}}^{T} \ (\underline{x},\underline{u}) \ . \tag{3.2}$$

Let $\underline{u}^*(t)$, $t_o \leq t \leq t_f$, be an admissible control such that the corresponding trajectory $\underline{x}(t)$ beginning at $\underline{x}_o$ at time t_o arrives at the terminal manifold T at some time t_f. In order that $\underline{u}^*(t)$ and $\underline{x}(t)$ be optimal, there must exist a non-zero continuous vector function $\underline{\lambda}(t)$ satisfying (3.2) such that for any t, $t_o \leq t \leq t_f$, the Hamiltonian function $H(\underline{x}(t), \underline{\lambda}(t), \underline{u}(t))$ considered as a function of the variable $\underline{u} \in U$ attains its maximum at the point $\underline{u} = \underline{u}^*(t)$, and that maximum value equals zero at every time t along the trajectory:

$$\max_{\underline{u} \in U} H(\underline{x}(t), \underline{\lambda}(t), \underline{u}(t)) = H(\underline{x}(t), \underline{\lambda}(t), \underline{u}^*(t)) = 0, \qquad t_o \leq t \leq t_f \ . \tag{3.3}$$

The maximum principle states that with each state point $\underline{x}$ in X is associated an adjoint vector $\underline{\lambda}$ corresponding to the optimal trajectory passing through $\underline{x}$. In the optimal system, then, $\underline{\lambda}$ can be considered to be a vector function of the state variables.

An additional necessary condition which must be satisfied at the terminus of every optimal trajectory is the transversality condition. Consider a trajectory $\underline{x}(t)$ which arrives at the terminal manifold T at the point $\underline{x}_f$ at time t_f. The q-dimensional hyperplane tangent to T at the point $\underline{x}_f$ is designated the tangent plane to T at $\underline{x}_f$. Then a necessary condition that $\underline{x}(t)$ be an optimal trajectory is that the vector function $\underline{\lambda}(t)$ corresponding to $\underline{x}(t)$, whose existence is assured by the maximum principle, is orthogonal to the tangent plane to T at the point $\underline{x}_f$, at the final time $t = t_f$.

A close relationship exists between the provisions of the maximum principle and the classical results of the calculus of variations. The system of 2n first-order differential equations (2.1) and (3.2) is equivalent to a system of n second-order Euler-Lagrange equations, while the maximum principle itself corresponds to the Weierstrass

Proofs of this result based on variational arguments have been given by Breakwell [Ref. 3] and by Berkovitz [Ref. 16]. A simplified derivation which also illustrates the geometry of the optimal control problem is presented here.

Consider the hypersurface $I(\underline{x}) = I_1$, where I_1 is a positive constant (Fig. 2), and denote this hypersurface T_1. This hypersurface consists of all the points $\underline{x}$ from which it is possible to attain the terminal manifold T with a cost of I_1, using optimal control. Let $\underline{x}^*(t)$ be the particular optimal trajectory emanating from an initial point $\underline{x}_o$ lying "outside" T_1, so that $I(\underline{x}_o) > I_1$, and let $\underline{x}_1$ be the point of intersection of $\underline{x}^*(t)$ with T_1.

Now the segment of $\underline{x}^*(t)$ connecting $\underline{x}_o$ and $\underline{x}_1$ is also the solution curve for a different optimal control problem, that of attaining the hypersurface T_1 from the point $\underline{x}_o$ with minimal cost; for if some other trajectory transferred the point $\underline{x}_o$ to T_1 with less cost than $\underline{x}^*(t)$, then since T can be attained from any point on T_1 with an optimal cost of I_1, the total cost from $\underline{x}_o$ to T could be made less than that obtained along the trajectory $\underline{x}^*(t)$, yielding a contradiction. Applying the transversality condition to this newly posed problem, for which T_1 now constitutes the terminal manifold, the adjoint vector $\underline{\lambda}$ corresponding to $\underline{x}^*(t)$ at the terminal point $\underline{x}_1$ must be normal to T_1 at $\underline{x}_1$, and is directed "inwards" towards T, since $\underline{\lambda}^T\underline{f} = \ell \geq 0$ at each point on any optimal trajectory. On the other hand, the gradient vector $I,_{\underline{x}}$ at the point $\underline{x}_1$ is normal to the hypersurface T_1 at $\underline{x}_1$ and is directed "outwards", in the direction of increasing $I(\underline{x})$. Hence, at the point $\underline{x}_1$,

$$\underline{\lambda} = -KI,_{\underline{x}}^T \quad , \tag{3.6}$$

where K is some positive constant.

To evaluate K, recall from the maximum principle that

$$H = \underline{\lambda}^T\underline{f} - \ell = -KI,_{\underline{x}}\underline{f} - \ell = 0 \tag{3.7}$$

at the point $\underline{x}_1$ on the optimal trajectory $\underline{x}^*(t)$. Now $I,_{\underline{x}}\underline{f}$ evaluated at $\underline{x}_1$ is just $\dot{I}$ at $\underline{x}_1$ along $\underline{x}^*(t)$. On the other hand,

$$I(\underline{x}_1) = \int_{t_1}^{t_f} \ell(\underline{x},\underline{u}^*)dt,$$

where t_1 is the time at which $\underline{x} = \underline{x}_1$ along $\underline{x}^*(t)$, and differentiation yields

$$\dot{I}(\underline{x}_1) = -\ell(\underline{x},\underline{u}^*(t_1)) .$$

Hence, (3.7) can be rewritten

$$\underline{H}(t_1) = -K\dot{I}(\underline{x}_1) + \dot{I}(\underline{x}_1) = 0,$$

and thus $K = 1$.

The above argument can be repeated for all points on the hypersurface $I(\underline{x}) = I_1$, and again for all other hypersurfaces $I(\underline{x}) = I_k$ where I_k is an arbitrary positive constant. Hence, the desired result (3.5) holds throughout X.

If the optimal cost function has only piecewise continuous first partial derivatives, the above arguments do not hold at points of discontinuity of these derivatives, since the gradient vector $I,_{\underline{x}}$ is not defined at such points. In such a case, (3.5) holds almost everywhere in X. In the sequel, where the result (3.5) is used, a tacit reservation to points of continuity of $I,_{\underline{x}}$ will be understood.

For any given problem the question of the satisfaction of the three initial assumptions arises. Unfortunately, the validity of the first two of these presuppositions is not subject to direct analytic test. Generally speaking, one's understanding of the nature of the particular problem under consideration will provide conviction of the correctness or incorrectness of these assumptions. The third assumption may often be established by investigation of the function $\ell(\underline{x},\underline{u})$, where the form of the control $\underline{u}$ is provided by application of the maximum principle.

Hence, if the optimal cost function $I(\underline{x})$ were known throughout X, the gradient vector $I,_{\underline{x}}$ could be written as a function of $\underline{x}$, and the sought-after optimal control law $\underline{c}^*(\underline{x})$ thereby obtained.

As a practical matter, however, non-optimal extremal trajectories may be feasibly distinguished simply by comparison. Thus, if the optimal solution is known to be a finite-time solution, i.e., if the state point arrives at the terminal manifold T at some finite time along any optimal trajectory, then if through every point in state space only one extremal trajectory passes, that extremal is, of necessity, optimal. If, however, through some set of state points more than one extremal trajectory passes, the true optimal trajectories can be distinguished from non-optimal ones by comparing the optimal cost functions along the separate contending trajectories at each such point.

On the other hand, it may not be known whether or not the optimal solutions arrive at the terminal manifold in finite time. For many problems, of course, such infinite-time solutions are clearly ruled out a priori, as in the time-optimal problem and problems where the final time is actually prescribed as a terminal condition. For some other problems - the linear system with a quadratic cost function, for example - the optimal trajectories are known to be infinite-time solutions [Ref. 19]. Clearly, such infinite-time solutions cannot be generated by any computational process such as those to be discussed below. However, such solutions may be approximated by solving the given problem with a specified final time chosen to be sufficiently large. Extremal solutions for the fixed-time problem may then be compared with any finite-time extremals of the original problem to determine those trajectories which are, in fact, truly optimal.

The generation of finite-time extremal trajectories will now be considered, assuming that the above-described methods can be used to distinguish among these trajectories those which are actually optimal.

In general, two distinct methods exist for generating extremal trajectories, both involving enforcement of the necessary conditions for optimality in the solution of the system equations. In the first method, which will not be discussed in any detail here, sequences of trajectories are generated from given initial conditions by forward integration of the system equations, with some technique for successive improvement ensuring that each trajectory in the sequence more closely equations can thus be written:

$$\dot{\underline{x}} = F\,\underline{x} + \underline{d}\,u \quad , \quad |u| \leq 1 \; . \tag{4.1}$$

The generalization to nonlinear systems, multidimensional control, and more complex control constraints will be obvious and straightforward.

1. The time-optimal problem; $q = 0$.

Here the terminal manifold is taken to be a point - in particular, the origin. For this problem $\ell = 1$ and the Hamiltonian function is

$$H = \underline{\lambda}^T(F\,\underline{x} + \underline{d}\,u) - 1.$$

Maximization of H with respect to $u \in U$ yields the optimal control law,

$$u^* = \operatorname{sgn} \underline{\lambda}^T\underline{d},$$

where "sgn" represents the signum function, defined by $\operatorname{sgn} f \equiv f/|f|$. The adjoint differential equations (3.2) are

$$\dot{\underline{\lambda}} = -\,F^T\,\underline{\lambda} \quad .$$

Since $H = 0$ everywhere on an extremal trajectory, $H = 0$ at $t = t_f$, in particular, where $\underline{x}(t_f) = \underline{0}$, and there results

$$H(t_f) = \underline{\lambda}_f^T\,\underline{d}\,\underline{u}^*_f - 1 = |\underline{\lambda}_f^T\,\underline{d}| - 1 = 0.$$

Since $q = 0$ there are no transversality requirements for this problem.

The extremal trajectories for this problem can be generated as solutions to the 2n-system of differential equations:

$$\begin{aligned} \dot{\underline{x}} &= F\,\underline{x} + \underline{d}\,\operatorname{sgn}(\underline{\lambda}^T\underline{d}) \; ; \; \underline{x}(t_f) = \underline{0} \\ \dot{\underline{\lambda}} &= -F^T\underline{\lambda} \qquad\qquad\quad ; \; \underline{\lambda}(t_f) = \underline{\lambda}_f, \text{ to satisfy } |\underline{\lambda}_f^T\underline{d}| = 1. \end{aligned} \tag{4.2}$$

where K is given by (4.3). There exists an (n-1)-parameter family of solutions to the system (4.4), corresponding to the n-1 arbitrary choices of components of $\underline{x}_f$, subject to the constraint $\underline{a}^T\underline{x}_f = 0$. Each of these solutions is an extremal for this problem.

3. Quadratic cost with fixed final time; $q = 0$.

Here, $\ell = \underline{x}^T Q\underline{x} + \gamma u^2$, where Q is a positive definite symmetric matrix and γ is a non-negative constant. The final time t_f is fixed. In order to properly represent the terminal manifold for this problem, an additional dynamic state is introduced, defined by the differential equation

$$\dot{x}_{n+1} = 1 \quad , \quad x_{n+1}(t_f) = t_f .$$

The terminal manifold in the (n+1)-dimensional state space is the point $(0,\ldots,0,t_f)$.

For this system,

$$H = \underline{\lambda}^T(F\underline{x} + \underline{d}u) + \lambda_t - \underline{x}^T Q\underline{x} - \gamma u^2 ,$$

where λ_t is the adjoint variable corresponding to the (n+1)th state variable. Application of the maximum principle yields the optimal control law:

$$u^* = \text{sat}\left(\frac{\underline{\lambda}^T\underline{d}}{2\gamma}\right) ,$$

where "sat" represents the saturation function, defined by

$$\text{sat } f \equiv \begin{cases} f & , \text{ if } |f| < 1 , \\ \text{sgn } f & , \text{ if } |f| \geq 1 . \end{cases}$$

The adjoint equations (3.2) are

The preceding examples have demonstrated the backwards generation of extremal trajectories as members of an (n-1)-parameter family of reversed-time solutions to a 2n set of first-order differential equations, the unknown parameters being components of the final state and adjoint vectors at the terminal manifold, in general. The disadvantage of this method is the difficulty in obtaining a set of values for these parameters which will insure that the distribution of the resulting extremals is reasonably uniform throughout X. Such a distribution is desirable since the overall accuracy of the data fitting procedure which follows depends on the uniformity of distribution of the data points. For some problems the physical significance of the components of the adjoint vector as sensitivities of the optimal cost function to changes in state may provide information of assistance in the appropriate selection of these parameter values. In general, however, several trial extremals must be generated from arbitrarily selected parameter values in order to find a set of values resulting in a suitable distribution of trajectories. However, since only a reasonable uniformity of spacing of trajectories is required, the evaluation of useful parameter values is not too critical a process, and the computation time needed for this preliminary exploration is generally only a fraction of that required for the synthesis procedure proper.

B. FUNCTIONAL APPROXIMATION OF THE OPTIMAL COST

After generating a suitable set of extremal trajectories by either the backward integration method described above or by some successive improvement technique, and after detecting and eliminating from consideration any extremal trajectories which are non-optimal, as discussed earlier, a functional approximation of the optimal cost function may be obtained in the following manner. First, the optimal cost I is tabulated as a function of the state variables $x_1,\ldots,x_n$ at regular increments of cost along each of the optimal trajectories. When these trajectories are obtained by backward integration, this tabulation is easily performed by integrating the function $\ell(\underline{x},\underline{u})$ along with the equations of motion from the initial condition $I = 0$ on the terminal for $i,j,\ldots,k = 0, 1,\ldots,m$. Introducing the definitions

$$T_{ij\ldots k} \equiv \sum_{\mu=1}^{N} I(\mu)\, x_1^i(\mu)\, x_2^j(\mu)\ldots x_n^k(\mu) \ ,$$

$$S_{ij\ldots k} \equiv \sum_{\mu=1}^{N} x_1^i(\mu)\, x_2^j(\mu)\ldots x_n^k(\mu) \ ,$$

for notational convenience, (4.8) can be rewritten as

$$\sum_{\alpha=0}^{m}\sum_{\beta=0}^{m}\cdots\sum_{\gamma=0}^{m} a_{\alpha\beta\ldots\gamma}\, S_{\alpha+i,\beta+j,\ldots,\gamma+k} = T_{ij\ldots k}; \quad \begin{array}{l} i=0,\ldots,m \\ j=0,\ldots,m \\ \vdots \\ k=0,\ldots,m. \end{array} \tag{4.9}$$

Equation (4.9) represents a system of $(m + 1)^n$ simultaneous linear equations in the $(m + 1)^n$ unknown coefficients $a_{\alpha\beta\ldots\gamma}$; $\alpha,\beta,\ldots,\gamma = 0, 1,\ldots,m$. Assuming that $N \geq (m + 1)^n$, it can be shown [Ref. 20] that the matrix of elements $S_{\alpha+i,\beta+j,\ldots,\gamma+k}$ is non-singular, guaranteeing a unique solution of (4.9) for the coefficients $a_{\alpha\beta\ldots\gamma}$ and thereby assuring a minimum for σ. Hence, solution of (4.9) by a suitably convenient computational method yields the coefficients of the approximation (4.7).

In calculating the elements of the S matrix and $\underline{T}$ vector, it will often be necessary to scale the state variables in order to avoid overflow or underflow conditions on the computer. A method for accomplishing this scaling is described in Appendix A.

When the order n of the dynamic system and the degree m of the approximating polynomial attain moderate size, the dimensionality of the system (4.9) becomes quite large. For example, with a fourth-order dynamic system and a fourth-degree approximation of $I(\underline{x})$, $n = 4$ and $m = 4$, and therefore there are $5^4 = 625$ simultaneous least-squares equations to solve. Thus the magnitude of the computing task increases quite rapidly as n and m increase. It is noted, however, that in is sought, where $p_i(x)$ is a polynomial of degree i, and where the coefficients $a_{ij...k}$ are again to be determined by the least-squares criterion. Following the reasoning which led to (4.9) above, there results an analogous system of linear equations,

$$\sum_{\alpha=0}^{m} \sum_{\beta=0}^{m} \cdots \sum_{\gamma=0}^{m} a_{\alpha\beta\ldots\gamma}\, S_{\alpha+i,\beta+j,\ldots,\gamma+k} = T_{ij\ldots k}; \quad \begin{matrix} i=0,\ldots,m \\ j=0,\ldots,m \\ \vdots \\ k=0,\ldots,m, \end{matrix} \tag{4.12}$$

where the elements of the $\underline{T}$ vector and the S matrix are given by

$$T_{ij\ldots k} \equiv \sum_{\mu=1}^{N} I(\mu) p_i(x_1(\mu)) p_j(x_2(\mu)) \ldots p_k(x_n(\mu))$$

$$S_{\alpha+i,\beta+j,\ldots,\gamma+k} \equiv \sum_{\mu=1}^{N} p_\alpha(x_1(\mu)) p_i(x_1(\mu)) p_\beta(x_2(\mu)) p_j(x_2(\mu)) \ldots \tag{4.13}$$

$$\ldots p_\gamma(x_n(\mu)) p_k(x_n(\mu)) .$$

If the polynomials $p_i(x)$ are chosen to be orthogonal with respect to some mass distribution $w(x)$ over some range (a,b), i.e., if

$$\int_a^b w(x)\, p_i(x)\, p_j(x) dx = 0, \qquad i \neq j,$$

then the off-diagonal elements of the S matrix, those elements for which $\alpha \neq i$, $\beta \neq j, \ldots, \gamma \neq k$, will generally be small in comparison with the diagonal terms. If the S matrix is so structured, with the diagonal elements dominating the off-diagonal ones in magnitude, then the accurate solution of the system (4.12) for the coefficients $a_{ij...k}$ is greatly facilitated, and approximations of relatively high degree can be obtained. As with the power series approximation, a considerable reduction in the dimensionality of the system (4.12) can be effected by fitting the reduced polynomial (4.11), with terms for which $(i+j+\ldots+k)>m$ eliminated.

a separate polynomial in each region of state space in which $I(\underline{x})$ is smooth. The controller would then have to contain logic to determine in which region of state space the current state is located, and to switch between appropriate sets of approximation coefficients whenever the state point crosses from one region to another.

Similar piecewise approximations might provide an attractive alternative when $I(\underline{x})$ is of such a nature that no approximating polynomial of conveniently low degree can be fitted with sufficient accuracy throughout the whole of X. Then suitable low order fits could be made locally in each of several subregions of X, and appropriate switching logic incorporated into the controller design.

where K is a positive scale factor, suitably chosen. A method for determining the value of K which has proven successful in yielding a reasonably accurate estimate of $\underline{\lambda}$ will be described.

Define the pseudo-Hamiltonian function,

$$\hat{H} \equiv \hat{\underline{\lambda}}^T \underline{f}(\underline{x},\underline{u}) - \ell(\underline{x},\underline{u}) = -K\hat{I},_{\underline{x}}\underline{f}(\underline{x},\underline{u}) - \ell(\underline{x},\underline{u}). \tag{5.3}$$

At each point $\underline{x}$, H differs from the Hamiltonian function H associated with the truly optimal solution in that (1) $\hat{\underline{\lambda}}$ is only an approximation to $\underline{\lambda}$, and (2) $\underline{u}$ is the non-optimal control resulting from inexact knowledge of $\underline{\lambda}$. The functional form of $\underline{u}$ is prescribed by application of the maximum principle, with the approximation $\hat{\underline{\lambda}}$ replacing $\underline{\lambda}$, Eq. (5.1). Now let K be chosen in accordance with another of the optimality conditions embodied in the maximum principle, namely the requirement that $\hat{H}$ vanish along the trajectory. Thus, at each point $\underline{x}$,

$$K = \frac{-\ell(x,u)}{\hat{I},_{\underline{x}}\underline{f}(\underline{x},\underline{u})} \quad , \tag{5.4}$$

where $\underline{u}$ is given by Eq. (5.1). This choice of K makes the pseudo-Hamiltonian a constant of the motion of the system, just as the true Hamiltonian is a constant of the optimal motion of the system. Of course, the third necessary condition for optimality, the requirement that $\hat{\underline{\lambda}}$ satisfy the adjoint differential equation (3.2), is not fulfilled, in general. Nevertheless, computational results indicate that the representation $\hat{\underline{\lambda}}$ of Eq. (5.2) with K given by (5.4) is a satisfactory approximation of $\underline{\lambda}$ for the purpose of control, the control law being given by (5.1).

B. EXAMPLES

To illustrate the control law calculation discussed above, two examples will be considered. In each case the approximate optimal control (5.1) will be found as a function of the state $\underline{x}$ and the computed

$$\hat{H} = -K\hat{I},_{\underline{x}}(F\underline{x} - \frac{K}{2\gamma}\underline{d}\,\hat{I},_{\underline{x}}\underline{d}) - K\hat{I},_{x_{n+1}} - \underline{x}^T Q\underline{x} - \frac{K^2(\hat{I},_{\underline{x}}\underline{d})^2}{4\gamma} =$$

$$\frac{(\hat{I},_{\underline{x}}\underline{d})^2}{4\gamma} K^2 - (\hat{I},_{\underline{x}}F\underline{x} + \hat{I},_{x_{n+1}})\,K - \underline{x}^T Q\underline{x} = 0.$$

Solving for K,

$$K = \frac{2\gamma}{(\hat{I},_{\underline{x}}\underline{d})^2}\left[\hat{I},_{\underline{x}}F\underline{x} + \hat{I},_{x_{n+1}} + \sqrt{(\hat{I},_{\underline{x}}F\underline{x} + \hat{I},_{x_{n+1}})^2 + \frac{\underline{x}^T Q\underline{x}}{\gamma}(\hat{I},_{\underline{x}}\underline{d})^2}\right],$$

where the positive sign before the radical was chosen to satisfy $K > 0$. The control law is thus given by

$$u = \text{sat}\left\{-\frac{1}{\hat{I},_{\underline{x}}\underline{d}}\left[\hat{I},_{\underline{x}}F\underline{x} + \hat{I},_{x_{n+1}} + \sqrt{(\hat{I},_{\underline{x}}F\underline{x} + \hat{I},_{x_{n+1}})^2 + \frac{\underline{x}^T Q\underline{x}}{\gamma}(\hat{I},_{\underline{x}}\underline{d})^2}\right]\right\}. \quad (5.6)$$

For the case where the final time is not prescribed, $\hat{I},_{x_{n+1}} \equiv 0$, and the control is otherwise identical to (5.6).

C. REALIZATION OF THE CONTROL LAW

The structure of the controller is dictated by the particular nature of the system being controlled and the cost function being minimized. The controller performs two specific computational functions:

1. It computes the gradient of the approximate optimal cost polynomial $\hat{I}(\underline{x})$, or at least those components of the gradient vector which are required in the determination of the control function. Each component is itself a polynomial in the state variables, and may either be computed directly from the coefficients of the approximation $\hat{I}(\underline{x})$ or from the derived coefficients of its own expansion. For example, the first component of the vector $\hat{I},_{\underline{x}}$ may be computed either as the polynomial

$$\hat{I},_{x_1} = \sum_{i=1}^{m}\sum_{j=0}^{m}\cdots\sum_{k=0}^{m} i\, a_{ij\ldots k}\, x_1^{i-1}\, x_2^{j}\ldots x_n^{k}\,, \quad (5.7)$$

VI. STABILITY

In the preceding chapter a control law $\underline{u} = \underline{c}(\underline{x})$ was obtained which, when combined with the system equations (2.1), yields the equations of motion for the controlled system:

$$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{c}(\underline{x})) \ . \tag{6.1}$$

In the development of this control law, no provision for insuring the stability of the resulting system has been made, where stability is here meant in the sense that all trajectories tend to the terminal manifold T as time increases. Of course, if T is a hypersurface in X, stable control is easily achieved, since it is only necessary that the state point intersect that hypersurface. However, for T of smaller dimension than n-1, the control law (5.1), even though approximately optimal, may generate trajectories which neither intersect T nor approach it asymptotically.

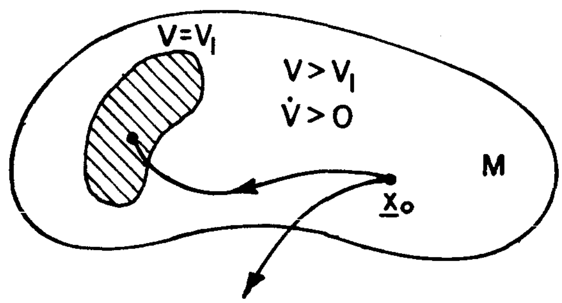
This problem is considered in the present chapter. First, the type of stability of interest here is defined precisely and sufficient conditions for guaranteeing stability are established. Then means of modifying the control in order to stabilize the system are discussed.

A. "STABILITY"

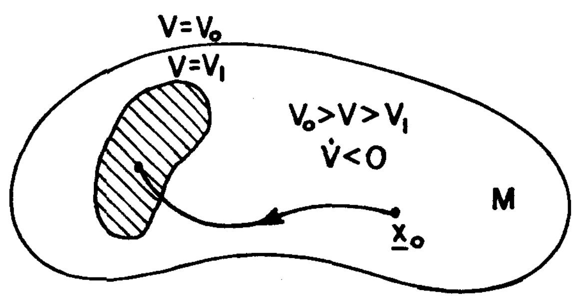
The subject of stability of a system of differential equations refers to the behavior of any solution curve with respect to a nearby known solution. By suitable substitution and transformation of coordinates the problem of stability is reducible to the question of behavior of the autonomous system

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \tag{6.2}$$

relative to the trivial solution, $\underline{x} = \underline{0}$ - the origin. Stability with respect to the origin is defined as follows [Ref. 22]: The origin is considerable value in the study of "stability" for the control problem being considered here. Let $V(\underline{x})$ be a continuous differentiable function defined in a region M such that $\dot{V}(\underline{x})$ along any trajectory in M is negative, or possibly zero at a countable number of points which are not equilibrium points of the system. Consider a hypersurface $V = V_1$ in M. It is clear that a trajectory starting at any $\underline{x}_o$ in M for which $V(\underline{x}_o) > V_1$ will eventually either intersect the hypersurface $V = V_1$ or will cross the boundary of M.



Further, if the boundary of M is in fact the hypersurface $V = V_o$ where $V_o > V_1$, then the trajectory starting at any point $\underline{x}_o$ for which $V_o > V(\underline{x}_o) > V_1$ cannot cross that boundary and therefore must eventually intersect the hypersurface $V = V_1$.



These ideas may be employed to establish a "stability" theorem analogous to the familiar Lyapunov theorems on stability.

from below in N $(V(\underline{x}) > K > 0,\ \underline{x} \in N)$. Further, T_δ is closed and bounded, and thus V is bounded from above in T_δ. Therefore, there exists some $\delta > 0$ small enough that $V(\underline{x}_2) > V(\underline{x}_1)$ for all $\underline{x}_1$ in T_δ and all $\underline{x}_2$ in N.

Let $\underline{x}(t)$ be a trajectory which at some time t_1 is in T_δ. Suppose at some later time $t_2 > t_1$ that $\underline{x}(t_2)$ is in N. This implies that $V(\underline{x}(t_2)) > V(\underline{x}(t_1))$ which is a contradiction, since $\dot{V} \leq 0$. Thus, if $\underline{x}(t)$ ever enters T_δ it never thereafter leaves T_ϵ, and quasistability of T is established.

Suppose $\underline{x}(t)$ neither crosses the boundary of M nor intersects T. Now $V(\underline{x})$ is continuous, bounded, positive and monotonic decreasing. Hence $\lim_{t \to \infty} V(\underline{x}(t)) = V_f \geq 0$ exists. Suppose $V_f > 0$. Then outside some region T_r, $-\dot{V}$ must tend to zero. But $-\dot{V} > 0$ in the closure of $M-T_r$, and since this region is bounded, $-\dot{V}$ has a positive minimum there, so that $-\dot{V}$ cannot in fact approach zero there. Thus $V_f = 0$, which implies that $\lim_{t \to \infty} d(\underline{x}(t)) = 0$. Hence $\underline{x}(t)$ is asymptotically quasistable.

This establishes the theorem.

B. STABILIZATION OF THE CONTROLLED SYSTEM

It is desired that the controlled system (6.1) be "stable". The theorem of the preceding section prescribes sufficient conditions for "stability"; hence its provisions may be employed to provide requirements whose satisfaction will ensure "stability" of the system. Specifically, a function $V(\underline{x})$ is sought for which the conditions of the theorem are fulfilled.

As a source of inspiration for a choice of $V(\underline{x})$ consider the optimal system. Of course this system is "stable", by definition, since every optimal trajectory necessarily attains the terminal manifold. Beyond this observation, however, it is easily seen that the optimal cost function $I(\underline{x})$ satisfies the requirements of the "stability" theorem. First, $I(\underline{x}) = 0$ on T and is positive everywhere else in X. Then along any optimal trajectory, by virtue of the relations interest here are those which originate in X, such a choice of M permits the possibility of trajectories leaving M to be ignored. Hence, satisfaction of the two conditions on $\hat{I}(\underline{x})$ stated above will, in fact, assure "stability".

In general, the approximation $\hat{I}(\underline{x})$, resulting as it does from fitting a polynomial to data points distributed throughout X, will not satisfy these two conditions. It might be possible to force these properties upon $\hat{I}(\underline{x})$ by applying these conditions as constraints on the fitting process; however, the quality of the overall fit would thereby necessarily be reduced and the system performance correspondingly degraded. Fortunately, satisfaction of these conditions on $\hat{I}(\underline{x})$ throughout X will not generally be required for system "stability". Since the control law (5.1) is nearly optimal, it can be assumed that all controlled trajectories will in fact arrive in the vicinity of T, and that therefore stabilization is required only in some neighborhood of T. This assumption is based on the observation that $\dot{\hat{I}}$ is negative along every controlled trajectory, and since $\hat{I}(\underline{x})$ is a reasonably good approximation to $I(\underline{x})$, its value is near zero in the vicinity of T and increases with increasing distance from T. Further, the value of the loss function $\ell(\underline{x},\underline{u})$ is generally small near T, so that the contribution of the terminal phase of the trajectory to the total cost is relatively insignificant. These considerations suggest the following control policy: the approximation $\hat{I}(\underline{x})$ resulting from the fitting procedure described in Chapter IV is to be used throughout X except in a suitably chosen neighborhood of T, where an approximation satisfying the two "stability" conditions stated above is to be substituted. Such a mode of operation will insure system "stability" while incurring only an insignificant cost penalty.

What sort of an approximation $\hat{I}(\underline{x})$ should be used near T for the purpose of stabilization? Any representation satisfying the two "stability" conditions would be satisfactory, of course, but an appropriately determined positive definite quadratic form possesses the virtues of simplicity and ease of implementation. The development of such an approximation will therefore be considered in more detail.

controller, is given by

$$\hat{I},_{\underline{x}} = 2\, \underline{y}^T Q\, \underline{y},_{\underline{x}} \;=\; 2(\underline{x} - \underline{t})^T Q \; .$$

In the most general case T is a manifold of arbitrary dimension. Define $\underline{y}(\underline{x})$ to be some convenient continuously differentiable vector function of $\underline{x}$ defined in X satisfying the condition that $\|\underline{y}(\underline{x})\| = 0$ if and only if $\underline{x} \epsilon T$. Thus $\|\underline{y}\|$ is a generalized measure of the distance of T from $\underline{x}$. In the vicinity of T let $\hat{I} = \underline{y}^T Q \underline{y}$, where again Q is positive definite to satisfy the "stability" conditions. Q can be determined by one of the methods described above. The gradient of the approximation is

$$\hat{I},_{\underline{x}} = 2\, \underline{y}(\underline{x})^T Q\, \underline{y}(\underline{x}),_{\underline{x}} \quad ,$$

for use in the computation of the control law (5.1).

The preceding paragraphs have described procedures for obtaining an approximation $\hat{I}(\underline{x})$ to guarantee "stability" in the vicinity of T. There remains the question of implementation of the transfer from the original approximation $\hat{I}(\underline{x})$ to the "stabilizing" approximation. In general the "stabilizing" approximation should be employed whenever the state point lies within a certain neighborhood of T, the boundaries of this region to be determined for the particular problem at hand on the basis of simulation results or experimentation. The mechanization of this switching may be performed in either of two ways.

1. The boundary hypersurface of the two regimes can be stored within the controller. One approximation is used when the state point lies on one side of this hypersurface, and the other used in the opposite case. This method will not be feasible if this switching boundary is highly convoluted.

2. The original approximation $\hat{I}(\underline{x})$ can be continuously computed by the on-line controller and the switch to the "stabilizing" approximation made whenever the value of $\hat{I}(\underline{x})$ falls below a prescribed level.

systems is completely automatic in its operation, and the efficacy of any particular procedure depends in part on the experience, skill and predilections of the person who is employing it.

For these reasons no direct comparative evaluation of various general synthesis techniques will be attempted. Instead, some observations will be made on those characteristics of the method proposed here which relate to its feasibility for practical application, characteristics such as required computational effort, accuracy, facility of control law mechanization, stability of control and system flexibility. Lacking definite knowledge about which general synthesis procedure is best for the particular application at hand, these are the factors which must be evaluated and weighed in order to come to a decision on which technique to employ.

Additionally in this chapter are discussed two extensions of the general synthesis procedure useful with certain classes of problems.

A. EVALUATION OF THE SYNTHESIS PROCEDURE

1. Computational Effort

As described in Chapter IV the proposed synthesis procedure requires an extensive amount of numerical calculation. This computation consists of the generation of a suitably large set of individual optimal trajectories, the evaluation and storage of the optimal cost function at several points along each of these trajectories, and finally the approximation of this function by a polynomial of appropriate degree in the n state variables by the method of least-squares fitting of the stored data points. Naturally these calculations are to be performed by a high-speed digital computer. In order that such a procedure be feasible, however, it must require neither an extensive amount of high speed memory, which is directly limited by current computer technology, nor excessive computation time, which is constrained less directly by economic considerations. In this section approximate computer time and storage requirements of the proposed synthesis method are indicated; primarily, these requirements are functions of the order n of the system and the degree m of the approximating polynomial.

$$L = (m+1)^n, \quad \text{or}$$

$$L = \binom{m+n}{m}$$

if the reduced fit of degree m is made.

On the basis of this second storage requirement then, for which low-speed storage cannot so readily be substituted without materially increasing computation time, the use of the reduced fit of moderate degree $(m = 4 \text{ or } 5)$ would permit systems of like order to be accommodated.

In order to make a rough estimate of the computation time required by this method, the procedure will be considered in three phases: (1) computation of optimal trajectories, assumed to be performed by the method of backwards integration; (2) calculation of the coefficients of the system of least-squares equations; (3) solution of the least-squares matrix equation.

Let Δ_1 be the average length of time required to numerically integrate a single differential equation backwards from the terminal manifold to the boundary of X. The magnitude of Δ_1 is determined primarily by the extent of the region X and the accuracy required of the integration procedure, which controls the integration step-size. Then the integration of each optimal trajectory will require an interval of length $2n\Delta_1$, on the average. Since τ^{n-1} such trajectories must be computed, the computation time for phase (1) is given by

$$2n \, \tau^{n-1} \, \Delta_1 \; .$$

Phase (2) requires the calculation of the coefficients of a system of L simultaneous linear algebraic equations, specifically the elements of the S and T matrices of (4.9). Since the matrix S is symmetric, $\frac{1}{2}L(L+3)$ such elements must be computed, each of which is a summation over the N observations, where N is given by (7.1). Each term of these summations is of the form $x_1^i \, x_2^j \ldots x_n^k$ or $I(x_1,\ldots,x_n)$ $x_1^i \, x_2^j \ldots x_n^k$. For simplicity, assume the average time to compute each uniform density of optimal trajectories in the synthesis procedure proper. The time used in generating these prefatory trajectories must also be included when figuring overall computer time requirements.

On the basis of the time estimates developed above it again appears that by using reduced fits of moderate degree, systems of equally moderate dimensionality (m,n up to four or five) may feasibly be treated by the proposed synthesis procedure in reasonable computation times.

2. Accuracy

At only one point in the synthesis procedure is approximation substituted for exact solution, and that, of course, is the approximation of the optimal cost function. Thus, if the polynomial $\hat{I}(\underline{x})$ were an exact representation of $I(\underline{x})$, the procedure would yield the true optimal control law. It may be concluded, then, that except for the small effects of computational errors such as roundoff and truncation which are inherent in any computer application, the inaccuracy of this functional approximation must be the sole cause of any discrepancies between the computed control law and the true optimal control. For this reason a consideration of the accuracy of the approximation $\hat{I}(\underline{x})$ will yield information about the accuracy of the resulting control law.

The question of accuracy of $\hat{I}(\underline{x})$ is inextricably bound up with the requirements on computer time and storage, as is evidenced by the repeated occurrence of the index m, the degree of the approximating polynomial, in the computer memory and time estimates of the preceding section. Given a computer of sufficient capacity and speed a very accurate representation of $I(\underline{x})$ could be obtained by generating sufficiently many data points and using an approximating polynomial of sufficiently high degree. In reality, however, computing capacity and time are limited, and rather severely so, as was indicated in the previous discussion. Hence, an appropriate question is: How accurately can $I(\underline{x})$ be represented by a polynomial of low or moderate degree?

No definite answer to this question can be expected, of course, since the response must vary with the nature of the particular problem being considered. Some remarks on the characteristics of the function

function. In fact, these components can be expected to vary considerably in sign and magnitude throughout X, and the components of c^* may possibly even be piecewise discontinuous. Therefore, these functions will not generally be susceptible to approximation by polynomials of moderate degree with an accuracy comparable to that attainable in approximating the optimal cost, $I(\underline{x})$. Furthermore, whereas only one approximation is required for $I(\underline{x})$, in estimating $\underline{\lambda}$ up to n scalar approximations must be made, and representation of $\underline{c}^*(\underline{x})$ requires r such approximations.

There exists another facet of the proposed method which influences the quality of the control. It would be convenient if the accuracy of the approximation $\hat{I}(\underline{x})$ were highest in those regions of X where the sensitivity of the optimal cost to changes in state is greatest, and where, therefore, deviations of the synthesized control from the true optimal control are most costly. No such regulation of the accuracy of approximation is operative in other synthesis methods. In the procedure proposed here, however, where data are stored at regular increments of cost along optimal trajectories, the density of tabulated points in state space is greatest in those regions where the cost changes most rapidly, i.e., where the sensitivity of the optimal cost is greatest. Then in the least-squares fitting procedure, there occurs a natural weighting of the fit in such regions because of the higher density of data points there. This results in a higher accuracy of approximation in just those regions where increased accuracy in the determination of the control law is most important. Unfortunately, no quantitative evaluation of this effect appears feasible.

The conclusion reached from the above discussion is that the nature of $I(\underline{x})$ is generally such that it can be adequately represented by a polynomial of reasonably low degree, while such other functions as $\underline{\lambda}(\underline{x})$ and $\underline{c}^*(\underline{x})$ cannot be similarly approximated with sufficient accuracy. Since the limitations of computer time and space considered in the preceding section dictate such an economy of approximation, the proposed procedure would appear to possess definite practical advantages over methods requiring such alternative approximations, particularly in quantization level is high, for accuracy of control, the controller storage requirement becomes excessive. A low quantization level, on the other hand, provides poor control accuracy, and if multivariate interpolation between tabulated values is planned in an effort to restore control quality, extensive on-line computation will be required. In short, the use of a tabulated control law appears feasible only with systems for which a reasonably large computer can be provided as an on-line controller.

4. Stability

Since any practical general synthesis method can yield only an approximation to the optimal control law, "stability" as defined in Chapter VI is never assured. Thus stabilization of the system in the vicinity of the terminal manifold becomes an additional requirement of any near-optimal synthesis procedure. Of course, such stabilization can generally be determined for the particular application at hand by some suitable design technique; the resulting mechanization is then incorporated into the controller for use in an operating mode which is independent of the near-optimal control law, to be employed solely for stabilization in a neighborhood of T. Such a procedure can undoubtedly provide the required "stability", but only at the price of an expanded design effort and an increased complexity in the controller mechanization.

On the other hand, it was seen in Chapter VI that the synthesis procedure being discussed here can provide stabilization with no alteration in the form of control from the approximate optimal control law, and hence with no modification in the design of the controller. Only the coefficients of the approximation $\hat{I}(\underline{x})$ need be changed. Furthermore, adjustment of these coefficients allows realization of a spectrum of compromise between near-optimal control and the requirements for "stability".

not appear explicitly in (7.2).) How does the presence of this constraint modify the computational procedure by which the control law is to be obtained?

The inclusion of such inequality constraints in the optimal control problem results in certain modifications to the necessary conditions for optimality [Refs. 1, 3, 16, 29, 30]. In general, during the interval when any optimal trajectory lies on the constraint boundary $\psi = 0$, the associated adjoint vector satisfies a differential equation different from (3.2), and in addition, the adjoint vector may be discontinuous at the point of first contact of the optimal trajectory with the constraint boundary.

These modifications necessitate certain changes in the synthesis procedure. In particular, the method of generating optimal trajectories must include provisions for satisfaction of the altered necessary conditions on the constraint boundary. Fortunately, these changes are readily implemented, whether the trajectories are obtained by backwards integration or by some successive improvement technique (see Ref. 6, for example).

The controller design may also require modification, depending on whether the constraint (7.2) is an actual "hard" constraint which is physically impossible for the system to violate, or a "soft" constraint mathematically prescribing operating limits which it is desired that the system not exceed. In the latter case, the controller must simulate the constraint boundary and insure that it not be violated.

2. Relay Control

Since the object of this investigation is the development of a general synthesis procedure applicable to a wide range of problems, no attention has been paid to the design of specific methods for handling special classes of problems. However, one type of problem encountered frequently appears particularly susceptible to treatment by a modification of the general synthesis procedure - a simplification, actually - and hence will be considered in some detail.

The simplest control mechanization requires the synthesis of the hypersurface of control discontinuity, usually termed the "switching surface", on each side of which the optimal control has different constant values. The determination of the equation of this hypersurface,

$$g(\underline{x}) = 0, \tag{7.4}$$

is thus the object of any synthesis method, and the control law to be realized by the on-line controller is simply

$$u = \text{sgn } g(\underline{x}) \ . \tag{7.5}$$

Exact determination of the switching surface has been accomplished for a few problems in which the system is linear and of low order (see, for example, Refs. 1, 2, 31). Generally, however, even when the nature of the switching surface is known, its accurate realization by an on-line controller is unfeasible, and hence approximations of the function $g(\underline{x})$ must be employed [Refs. 2, 31, 32, 33]. For the general relay control problem, such an approximation of the switching surface can be conveniently incorporated into the proposed synthesis procedure in the following manner.

The optimal trajectories are generated just as described in Chapter IV, but the only data stored are the values of the n state variables at each switch point, where a control discontinuity occurs. Each such state point lies on the switching surface, and therefore a representation of that surface can be obtained by least-squares fitting of the polynomial approximation

$$\hat{x}_\alpha = \sum_{i=0}^{m} \sum_{j=0}^{m} \cdots \sum_{k=0}^{m} a_{ij\ldots k} \, x_1^i \, x_2^j \cdots x_{\alpha-1}^{\mu} \, x_{\alpha+1}^{\nu} \cdots x_n^k \tag{7.6}$$

to the set of these data points, where α can take on any value from 1 to n. In general, the accuracy of the approximation will depend on which index is chosen as α, so that for the best approximation, n

VIII. EXAMPLES

This investigation concludes with an account of the application of the synthesis procedure described in these pages to some representative optimal control problems. The consideration of these examples in some detail is intended to fulfill the following three objectives: (1) to substantiate the correctness and validity of the method as a synthesis technique; (2) to support the arguments and evaluations put forth in the preceding chapter relative to the practicality and applicability of the procedure,which in the absence of corroborating examples would remain largely conjectural; and (3) to illustrate by specific example the operation of the method and thereby to elucidate details of its application to actual problems, where heretofore the procedure has been described primarily in general terms.

To provide the means for treating such problems, a program comprising an implementation of the synthesis procedure and a simulation of the resulting controlled system was written in FORTRAN for the IBM 7090. The capabilities of this program include the generation of optimal trajectories, either by the method of backwards integration or by a gradient optimization procedure [Ref. 5], the calculation and storage of the optimal cost function $I(\underline{x})$ at points along these trajectories, and the approximation of $I(\underline{x})$ by a polynomial of arbitrary degree by the method of least-squares fitting to the stored data points. A simplified flow diagram of the program is included as Appendix B. The program was used to synthesize near-optimal feedback control laws for the three examples to be discussed in this chapter, as well as to simulate the resulting controlled systems for the purpose of evaluation and verification of the synthesis procedure.

The three chosen examples display several diverse aspects and characteristics representative of many different kinds of optimal control problems; included are cases of linear and non-linear systems, single and multivariable controls, continuous and bang-bang controls, terminal manifolds of varying dimensionality, infinite-time optimal solutions, piecewise approximation of $I(\underline{x})$, switching surface

The solution to the general linear problem when the control variable u is unconstrained is known [Ref. 19]. The optimal control is of the form

$$u^* = -\frac{1}{\gamma}\underline{d}^T P\underline{x} , \tag{8.3}$$

involving linear feedback of the state variables, where the matrix P is the steady-state solution of the matrix Riccati equation

$$\dot{P} = Q + PF + F^T P - \frac{1}{\gamma} P\underline{dd}^T P \tag{8.4}$$

obtained by setting the steady-state $\dot{P}$ to $[0]$ and solving (8.4) for the elements of P. The optimal cost function is simply the quadratic form

$$I(\underline{x}) = \underline{x}^T P\underline{x} . \tag{8.5}$$

For the particular system of Fig. 3, the matrix P is computed to be

$$P = \begin{bmatrix} \sqrt{2(2\sqrt{2}-1)} & -1+\sqrt{2} \\ -1+\sqrt{2} & \sqrt{2\sqrt{2}-1} \end{bmatrix} = \begin{bmatrix} 1.912 & 0.414 \\ 0.414 & 1.352 \end{bmatrix} . \tag{8.6}$$

The optimal control law for this example with unconstrained control, from (8.3), is

$$u^* = -0.414x_1 - 1.352x_2 . \tag{8.7}$$

It will be noted that in this optimal system, the state variable histories are exponentially damped sinusoidal functions, and thus the origin is not attained in finite time.

$$u^* = \text{sat}(\tfrac{1}{2}\lambda_2), \tag{8.8}$$

and the differential system (4.6) to be integrated becomes

$$\begin{aligned}
\dot{x}_1 &= x_2 &;\quad& x_1(10) = 0 \\
\dot{x}_2 &= -x_1 + \text{sat}(\tfrac{1}{2}\lambda_2) &;\quad& x_2(10) = 0 \\
\dot{t} &= 1 &;\quad& t(10) = 0 \\
\dot{\lambda}_1 &= \lambda_2 + 2x_1 &;\quad& \lambda_1(10) = \lambda_{1_f} \\
\dot{\lambda}_2 &= -\lambda_1 + 2x_2 &;\quad& \lambda_2(10) = \lambda_{2_f} \\
\dot{\lambda}_t &= 0 &;\quad& \lambda_t(10) = \lambda_t ,
\end{aligned} \tag{8.9}$$

where λ_t (4.5) is given by

$$\lambda_t = \begin{cases} 1 - |\lambda_{2_f}| & , \quad |\lambda_{2_f}| \geq 2 \\ -\tfrac{1}{4}\lambda_{2_f}^2 & , \quad |\lambda_{2_f}| < 2 . \end{cases} \tag{8.10}$$

The terminal values λ_{1_f} and λ_{2_f} are arbitrary. In this problem the system equations (8.9) are to be integrated backwards in time until $t = 0$, at which instant the states x_1 and x_2 and the associated cost will be recorded. Hence, values of λ_{1_f}, λ_{2_f} which result in a reasonably uniform distribution of points $\underline{x}(0)$ throughout X are desired. A trial set of 169 trajectories was generated from the grid of terminal values: $\lambda_{1_f}, \lambda_{2_f} = 0, \pm 0.0001, \pm 0.0005, \pm 0.001, \pm 0.005, \pm 0.01, \pm 0.02$. On the basis of these results, a final selection of terminal conditions was made: $\lambda_{1_f}, \lambda_{2_f} = 0, \pm 0.0015, \pm 0.003, \pm 0.005, \pm 0.008, \pm 0.013, \pm 0.02$. For each combination of values for λ_{1_f} and λ_{2_f}, the system (8.9) was integrated backwards in time until $t = 0$. (In practice, neither the 3rd nor 6th equation of (8.9) actually required integration, since the variable t is always available as the independent

After generating the set of optimal trajectories, five approximations of the optimal cost function for this modified problem were obtained as polynomials of the form

$$\hat{I}(\underline{x}) = \underbrace{\sum_{i=0}^{m} \sum_{j=0}^{m}}_{i+j \leq m} a_{ij} \, x_1^i \, x_2^j \tag{8.11}$$

by making reduced least-squares fits of degree $m = 2,\ldots,6$ to the 169 stored data points. The root-mean-square errors of these five approximations over the set of observations were

$$\epsilon_2 = 0.4697$$

$$\epsilon_3 = 0.4711$$

$$\epsilon_4 = 0.1797$$

$$\epsilon_5 = 0.2184$$

$$\epsilon_6 = 0.2536,$$

where the average optimal cost over the data points had an approximate value of 8.4. The slight discrepancy in the value of ϵ_3 is attributable to numerical eccentricities occasioned by the process of fitting a polynomial of odd degree to a set of data points which are completely symmetric about the origin, while the values of ϵ_5 and ϵ_6 show the effect of increasing numerical inaccuracy in the solution of the least-squares system (4.9) as the condition of singularity is approached by the matrix S with increasing m. On the basis of these results, the approximation of degree four was selected as the best representation of $I(\underline{x})$. The coefficients of this approximation are listed in Table 1.

$$\hat{I}_{,x_1} = \underbrace{\sum_{i=1}^{4} \sum_{j=0}^{4}}_{i+j \leq 4} i\, a_{ij}\, x_1^{i-1}\, x_2^{j} \quad ,$$

$$\hat{I}_{,x_2} = \underbrace{\sum_{i=0}^{4} \sum_{j=1}^{4}}_{i+j \leq 4} j\, a_{ij}\, x_1^{i}\, x_2^{j-1} \quad . \tag{8.13}$$

As indicated in Chapter VI, application of the control law (8.12) does not insure "stability", which for this example is equivalent to asymptotic stability, since T is the origin, an equilibrium point for the system. Stability can be guaranteed, however, by substituting for the fourth-degree approximation (8.11) a quadratic form

$$\hat{I}(\underline{x}) = \underline{x}^T R\underline{x} \ , \tag{8.14}$$

R being a positive definite square matrix, in some neighborhood of the origin. Though any qualified R will provide stabilization, the analytic results stated at the beginning of this example for the case with unconstrained control show that in the vicinity of the origin stability and optimality can be combined by selecting $R = P$, where P is given by (8.6). Then the approximation (8.14) is exact and the resulting control (8.12) is truly optimal, being equivalent to the control given by (8.7). Of course, best results will be obtained when the representation (8.14) is employed throughout the entire neighborhood of the origin in which the optimal constrained control never subsequently saturates, but application of such a policy requires previous knowledge of the limits of this region. In the absence of this information a more arbitrary determination of the approximation-switching boundary must be made.

If the analytic results for the unconstrained control problem were not known, a logical method for selecting the matrix R would be to eliminate all but the quadratic terms of the complete polynomial approximation (8.11), as suggested in Chapter VI. When this is done for the

TABLE 2. RESULTS OF SIMULATION OF THE CONTROLLED SYSTEM OF EXAMPLE A.

TRAJECTORY	INITIAL CONDITIONS		COST		COST PENALTY:
	$x_1(o)$	$x_2(o)$	Optimal	Using Synthesized Control	Percentage Above Optimal Cost
1	-2.358	0.802	10.658	10.662	0.04
2	-2.305	1.486	12.368	12.371	0.02
3	-1.846	2.013	11.959	11.965	0.05
4	-1.293	2.445	12.815	12.834	0.15
5	-0.665	2.867	16.118	16.142	0.15
6	0.457	3.075	20.755	20.758	0.01
7	0.704	2.841	18.293	18.295	0.01
8	1.172	2.387	15.309	15.313	0.03
9	1.764	1.623	12.798	12.803	0.04
10	2.073	1.026	11.839	11.843	0.03
11	2.301	0.110	10.715	10.719	0.04

B. EXAMPLE B

Consider the single-axis model of a space vehicle shown in Fig. 4. Stabilization of the inertial attitude of this vehicle in the presence of both impulsive perturbations and constant disturbing torque is required. The control actuating devices include an inertia wheel driven by a d-c motor for momentum storage and a pair of gas jets for momentum dumping. Assuming these actuators to be located symmetrically with respect to the vehicle's principal axis, the equation of attitude motion can be written:

$$I\ddot{\theta} - J\dot{\Omega} = T_J + T_D \quad , \quad |T_J| \le T_{J_m} \quad , \tag{8.16}$$

The cost function to be minimized in the control process is a weighted sum of the fuel expended by the gas jets and the integral squared attitude error:

$$\ell = \left|\frac{T_J}{T_{J_m}}\right| + \gamma\,\theta^2 , \tag{8.18}$$

γ being an arbitrary positive weighting factor.

The following state variable and control variable definitions are made:

$$
\begin{aligned}
x_1 &\equiv \theta \\
x_2 &\equiv \dot{\theta} \\
x_3 &\equiv \frac{\Omega}{\Omega_m} \\
u_1 &\equiv Ke \quad , \quad (|u_1| \leq u_{1_m} \equiv Ke_m) \\
u_2 &\equiv \frac{T_J}{I} \quad , \quad (|u_2| \leq u_{2_m} \equiv \frac{T_{J_m}}{I}).
\end{aligned}
$$

Then the system equations (8.16) and (8.17) can be rewritten in the form (2.1):

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -u_{1_m}\,\beta x_3 + \beta u_1 + u_2 + v \\
\dot{x}_3 &= -\alpha\,x_3 + \frac{\alpha}{u_{1_m}}\,u_1 ,
\end{aligned}
\tag{8.19}
$$

where $v \equiv T_D/I$ and $\beta \equiv J/I$. The loss function becomes

$$\ell(\underline{x},\underline{u}) = \left|\frac{u_2}{u_{2_m}}\right| + \gamma\,x_1^2 . \tag{8.20}$$

The linear appearance of u_1 in the Hamiltonian suggests the possibility of singular optimal control, which must be investigated [Refs. 17, 18]. Examination of the problem shows that the only singular optimal trajectory which exists coincides with the x_3-axis between the limits $x_3 = -1 + v/\beta u_{1_m}$ and $x_3 = 1 + v/\beta u_{1_m}$. Thus, when $x_1 = x_2 = 0$ and x_3 lies in this region, the optimal trajectory is singular, lying along the x_3-axis. The optimal control in this region is $u_1 = u_{1_m} x_3 - v/\beta$, $u_2 = 0$. Since the object of the control action in this example is to attain the x_3-axis from elsewhere in X, this singular solution is of no further concern.

TABLE 3. SYSTEM PARAMETER VALUES FOR EXAMPLE B.

Vehicle and Motor Parameters		Derived System Parameters	
$I = 150.$	slug-ft^2	$\alpha = 1/\tau = 0.025$	sec^{-1}
$J = 0.002$	slug-ft^2	$\beta = J/I = 1.333 \times 10^{-5}$	
$T_{J_m} = 0.1$	ft-lb	$u_{1_m} = \Omega_m/\tau = 10.$	sec^{-2}
$\tau = 40.$	sec	$u_{2_m} = T_{J_m}/I = 0.667 \times 10^{-3}$	sec^{-2}
$\Omega_m = 400.$	rad/sec	$v = T_D/I = 0.5 \times 10^{-4}$	sec^{-2}

Application of the maximum principle yields the optimal control law:

$$u_1^* = u_{1_m} \operatorname{sgn} \left(\beta\lambda_2 + \frac{\alpha}{u_{1_m}} \lambda_3\right)$$

$$u_2^* = \begin{cases} 0 & ; \; u_{2_m} |\lambda_2| \leq 1 \\ u_{2_m} \operatorname{sgn}\lambda_2 & ; \; u_{2_m} |\lambda_2| > 1 . \end{cases} \tag{8.22}$$

Thus the nature of the optimal control voltage to the d-c motor is bang-bang, while the optimal gas jet control is bang-off-bang.

both must be employed in order that the complete family of extremal solutions be generated. If v lies outside the range indicated, the system cannot be controlled.

Approximately 50 exploratory trajectories were generated using various combinations of values for the two arbitrary parameters x_{3_f} and λ_{1_f}. On the basis of those results a grid of 41 pairs of values for these terminal conditions was selected to yield a reasonably uniform distribution of trajectories throughout X. These selections are displayed in Table 4. For each chosen pair of parameters, equations (8.23) were integrated backwards from the x_3-axis to the boundary of X, and the loss function (8.20) was simultaneously integrated to obtain the optimal cost.

TABLE 4. TERMINAL PARAMETER VALUES FOR SELECTED OPTIMAL TRAJECTORIES OF EXAMPLE B.

x_{3_f}	λ_{1_f}	x_{3_f}	λ_{1_f}	x_{3_f}	λ_{1_f}
-0.75	0	-0.25	0	0.25	0
-0.75	0.10×10^{-3}	-0.25	0.20×10^{-3}	0.25	0.20×10^{-2}
-0.75	0.20×10^{-3}	-0.25	0.20×10^{-2}	0.25	0.50×10^{-2}
-0.75	0.20×10^{-2}	-0.25	0.50×10^{-2}	0.25	0.70×10^{-2}
-0.75	0.60×10^{-2}	-0.25	0.10×10^{-1}	0.25	0.10×10^{-1}
-0.75	0.15×10^{-1}	-0.25	0.20×10^{-1}	0.25	0.30×10^{-1}
-0.75	0.50×10^{-1}	-0.25	0.50×10^{-1}	0.25	0.70×10^{-1}
-0.50	0	0	0	0.50	0
-0.50	0.10×10^{-3}	0	0.20×10^{-2}	0.50	0.10×10^{-3}
-0.50	0.20×10^{-3}	0	0.50×10^{-2}	0.50	0.50×10^{-2}
-0.50	0.20×10^{-2}	0	0.70×10^{-2}	0.50	0.70×10^{-2}
-0.50	0.60×10^{-2}	0	0.10×10^{-1}	0.50	0.10×10^{-1}
-0.50	0.15×10^{-1}	0	0.30×10^{-1}	0.50	0.20×10^{-1}
-0.50	0.50×10^{-1}	0	0.70×10^{-1}		

In this problem the optimal control u_1 can take on two possible values, $\pm u_{1_m}$, while u_2 can assume three, $\pm u_{2_m}$ or 0. Hence six

$$\hat{I}(x_1,x_2,x_3) = \underbrace{\sum_{i=0}^{4} \sum_{j=0}^{4} \sum_{k=0}^{4}}_{i+j+k \leq 4} a_{ijk}\, x_1^i\, x_2^j\, x_3^k \quad , \tag{8.25}$$

are listed in Table 5 for each of these regions.

The computer time required for the integration of the 50 exploratory trajectories was about five minutes, while the synthesis procedure proper, comprising the generation of 41 trajectories and the fitting of a fourth-degree polynomial to the stored data in each of four separate control regions, required 12.8 min., so that the 7090 was occupied with the solution of this example less than 20 minutes altogether.

The control law for this example is obtained by the method described in Chapter V. The motor control is given by

$$u_1 = -u_{1_m} \operatorname{sgn}(u_{1_m} \beta\, \hat{I},_{x_2} + \alpha\, \hat{I},_{x_3}). \tag{8.26}$$

Define

$$\xi \equiv -\hat{I},_{x_1} x_2 + (u_{1_m} \beta\, \hat{I},_{x_2} + \alpha\, \hat{I},_{x_3})\,(x_3 - \operatorname{sgn} u_1) - \hat{I},_{x_2} v.$$

The gas jet control is then expressed as

$$u_2 = \begin{cases} 0 & ;\ \gamma u_{2_m} x_1^2\, |\hat{I},_{x_2}| \leq \xi \\ -u_{2_m} \operatorname{sgn} \hat{I},_{x_2} & ;\ \gamma u_{2_m} x_1^2\, |\hat{I},_{x_2}| > \xi \quad . \end{cases} \tag{8.27}$$

To make use of the piecewise approximations of $I(\underline{x})$, the controller must also possess the capability of tracking the state point from one control region to another in order to insure that the approximation being employed at each instant is the correct one.

TABLE 5. (Con'd) COEFFICIENTS a_{ijk} OF THE FOURTH-DEGREE PIECEWISE APPROXIMATIONS TO THE OPTIMAL COST FUNCTION FOR EXAMPLE B.

INDICES			REGION 3	REGION 4
i	j	k	$(u_1 = u_{1_m},\ u_2 = u_{2_m})$	$(u_1 = -u_{1_m},\ u_2 = -u_{2_m})$
0	0	0	-0.3053776E 01	-0.1789962E 01
0	0	1	-0.3133815E 01	-0.6636929E 01
0	0	2	0.4470305E 01	0.2144950E-00
0	0	3	0.1513141E 01	0.1748507E 01
0	0	4	-0.8813556E 00	0.3334343E-00
0	1	0	-0.9624212E 03	0.2074296E 04
0	1	1	-0.2387743E 04	0.3572028E 03
0	1	2	0.8596217E 03	-0.9656503E 03
0	1	3	0.1046089E 03	-0.1632849E 03
0	2	0	0.1617800E 06	0.1664989E 05
0	2	1	-0.3809505E 06	0.1456934E 06
0	2	2	0.5434729E 05	0.9333789E 05
0	3	0	0.1730506E 08	-0.1858851E 08
0	3	1	-0.2256076E 08	-0.1853190E 08
0	4	0	0.1425688E 10	0.2315936E 10
1	0	0	-0.9769628E 02	0.3151815E 02
1	0	1	0.1067342E 02	0.6585301E 02
1	0	2	0.5482717E 01	-0.6999390E 01
1	0	3	0.3721382E 01	-0.2869626E 01
1	1	0	-0.1203975E 05	0.2056817E 04
1	1	1	-0.1211287E 05	-0.9526722E 04
1	1	2	0.3367510E 03	0.4040602E 04
1	2	0	-0.3016021E 07	0.9570177E 06
1	2	1	-0.2061537E 07	-0.6683209E 06
1	3	0	0.1142644E 08	0.1946525E 09
2	0	0	0.8247879E 03	0.1505690E 04
2	0	1	-0.2914464E 03	-0.3423422E 03
2	0	2	-0.1000811E 03	0.1129085E 03
2	1	0	-0.2312255E 06	0.4933394E 05
2	1	1	-0.1123158E 06	0.6406429E 04
2	2	0	-0.1010241E 08	0.7652873E 07
3	0	0	-0.3243907E 04	-0.1207635E 04
3	0	1	-0.3170773E 04	0.4227084E 03
3	1	0	-0.5651860E 06	0.3084269E 06
4	0	0	-0.1070744E 05	0.6706210E 04
RMS error			0.258	0.615

TABLE 6. RESULTS OF SIMULATION OF THE CONTROLLED SYSTEM OF EXAMPLE B.

TRAJ.	INITIAL CONDITIONS			OPTIMAL COST	SYNTHESIS USING APPROXIMATION TO $I(\underline{x})$		SYNTHESIS USING APPROXIMATION TO SWITCHING SURFACES	
	$x_1(0)$ (rad)	$x_2(0)$ (rad/sec)	$x_3(0)$		Cost	Pct. Above Opt. Cost	Cost	Pct. Above Opt. Cost
1	-0.2054	0.0200	0.7376	55.88	56.15	0.48	56.56	1.22
2	0.1619	-0.0200	0.1369	49.81	51.00	2.39	50.03	0.44
3	-0.1607	0.0200	0.9778	47.32	47.52	0.42	47.33	0.02
4	0.2140	-0.0207	0.2120	50.19	50.61	0.84	50.20	0.02
5	0.2008	-0.0177	0.3668	38.97	39.42	1.16	39.05	0.21
6	-0.1196	-0.0042	1.0746	44.51	45.54	2.32	46.03	3.42
7	0.1772	-0.0203	-0.3937	41.57	43.33	4.23	41.72	0.36
8	-0.2087	0.0204	-0.5497	77.65	77.78	0.17	77.65	0.00
9	0.1579	0.0003	-1.2021	46.31	46.64	0.71	47.23	1.99

Using this control, the equations of motion were integrated forwards from the same nine initial conditions used before, and the results of these simulations are also included in Table 6. The control accuracy achieved by the two control laws is seen to be of comparable quality, but of course the implementation of (8.31) in an on-line controller is far simpler than the mechanization of (8.26) and (8.27) together with the associated tracking and switching logic. The computation time required by the second synthesis method is also less than for the first.

C. EXAMPLE C

Figure 7 depicts a planar space rendezvous problem in which a pursuing vehicle P is to be brought to a stationary position alongside a target vehicle T. Control of P is effected by a throttleable, steerable rocket motor. The equations of motion for this system are

$$\begin{aligned} \ddot{r} - r\dot{\gamma}^2 &= a \cos \theta \\ r\ddot{\gamma} + 2\dot{r}\dot{\gamma} &= a \sin \theta \ , \end{aligned} \tag{8.32}$$

where r is the range to P from T, $\dot{\gamma}$ is the angular velocity of the line-of-sight to P from T, a is the acceleration of P, assumed controllable between the limits zero and a_m, and θ is the angle of the axis of the rocket motor relative to the line-of-sight TP, as shown in Fig. 7.

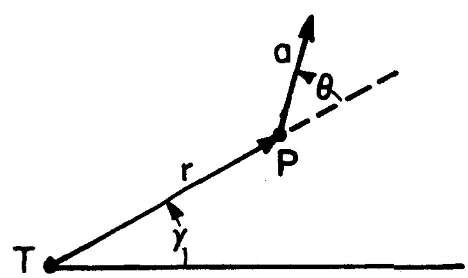


FIG. 7. THE SYSTEM OF EXAMPLE C.

A set of nominal initial conditions for the rendezvous maneuver is assumed:

$$x_1(t_o) = r_o$$

$$x_2(t_o) = \dot{r}_o$$

$$x_3(t_o) = 0 \ .$$

The control system is to be designed to provide near-optimal control along any trajectory with initial conditions in a given neighborhood of the nominal set. In particular, at the nominal initial range of r_o, off-nominal conditions of range rate and line-of-sight rate in the regions

$$\dot{r}_o - \Delta_2 \leq x_2(t_o) \leq \dot{r}_o + \Delta_2$$

$$- \Delta_3 \leq x_3(t_o) \leq \Delta_3$$

are to be accommodated. This specification implicitly defines the region X for this example.

The numerical values chosen for the parameters of this problem comprise Table 8. The values of Δ_2 and Δ_3 correspond to initial errors in the relative velocity of P of up to 50 ft/sec in magnitude and 20^o in direction at the nominal initial range.

TABLE 8. SYSTEM PARAMETER VALUES FOR EXAMPLE C

$a_m = 0.5$	$r_o = 100{,}200.$ ft
$K_1 = 0.0069444$	$\dot{r}_o = -200.$ ft/sec
$K_2 = 10{,}000$	$\Delta_2 = 46.9845$ ft/sec
$r_f = 200.$ ft	$\Delta_3 = 17.067\text{x}10^{-5}$ rad/sec

and where the values of λ_{1_f} and λ_{3_f} are arbitrary.

110 exploratory trajectories were generated by trying various combinations of values for the parameters λ_{1_f} and λ_{3_f}. Based on these results, a set of 81 pairs was selected to yield a reasonably uniform distribution of optimal trajectories. These selections are listed in Table 9. For each pair, the corresponding optimal trajectory was generated by integrating (8.37) backwards from the terminal manifold to the plane $x_1 = 125{,}000$ ft. Along each trajectory the state and associated cost were tabulated at cost increments of ten. In all, 572 data points were obtained.

To avoid numerical problems of overflow and underflow in the approximating procedure because of the range of values assumed by the state variables of this example, it was decided to scale these variables for the least-squares fitting process (Appendix A). Accordingly, the variables $y_i = k_i x_i$, $i = 1,2,3$ were stored instead of the states themselves, where the scale factors chosen were $k_1 = 10^{-4}$, $k_2 = 1$, $k_3 = 10^4$.

Three polynomial approximations of $I(\underline{y})$ were obtained by making reduced fits of degrees two, three and four to the 572 data points. The root-mean-square errors of these approximations over the data were

$$\epsilon_2 = 3.4685$$

$$\epsilon_3 = 0.4453$$

$$\epsilon_4 = 0.3543,$$

where the average value of I was approximately 46.2. The coefficients of the fourth-degree approximation, the one selected for use in the controlled system simulation, are listed in Table 10.

The computer time required in generating the 110 exploratory trajectories was 7.8 minutes, while the synthesis procedure proper required 13.0 minutes.

TABLE 10. COEFFICIENTS a_{ijk} OF THE FOURTH-DEGREE APPROXIMATION TO THE OPTIMAL COST FUNCTION FOR EXAMPLE C.

INDICES i	j	k	COEFFICIENT
0	0	0	0.2555248E 01
0	0	1	-0.2326450E-02
0	0	2	0.1845993E-01
0	0	3	-0.4155344E-05
0	0	4	-0.2048923E-03
0	1	0	0.2096336E-00
0	1	1	-0.1485633E-03
0	1	2	0.5372780E-04
0	1	3	-0.8236343E-06
0	2	0	0.1165180E-01
0	2	1	-0.3973867E-05
0	2	2	0.2402931E-05
0	3	0	0.4485767E-04
0	3	1	-0.5631363E-07
0	4	0	0.1002181E-06
1	0	0	0.6806199E 01
1	0	1	-0.1881683E-02
1	0	2	0.1321589E-01
1	0	3	-0.1705066E-04
1	1	0	0.4050164E-00
1	1	1	-0.1020002E-03
1	1	2	-0.3272329E-04
1	2	0	0.9765793E-03
1	2	1	-0.2861653E-05
1	3	0	0.3248978E-05
2	0	0	0.3348927E 01
2	0	1	-0.5767450E-03
2	0	2	0.1000166E-02
2	1	0	-0.1873337E-01
2	1	1	-0.4982913E-04
2	2	0	0.3480404E-04
3	0	0	-0.3715040E-00
3	0	1	-0.2983591E-03
3	1	0	0.9839372E-03
4	0	0	0.1471258E-01

compute a new approximation of $I(\underline{x})$ for use in the vicinity of T, a simple alternate control scheme was sought which would permit simulation of the terminal phase of each trajectory.

Examination of the optimal trajectories generated in the synthesis procedure indicated that in the final portion of each trajectory the value of the optimal control u_1 decreases almost linearly with time to the terminal value $u_1(t_f) \cong 1/12$. Furthermore, the angular velocity x_3 is reduced essentially to zero early along each trajectory, so that the terminal phase is closely described by the simple second-order linear system resulting from (8.34) by setting x_3 to zero:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u_1 \;. \end{aligned} \tag{8.40}$$

On the basis of these observations, then, the following simple terminal control scheme was provided in the simulation. Let t_o, x_{1_o} and x_{2_o} denote the values of time and the states x_1, x_2 at the point at which the terminal control scheme is initiated. The control u_1 is then obtained in the form

$$u_1(t) = u_{1_o} - \left(\frac{u_{1_o} - \frac{1}{12}}{t_f - t_o} \right) (t - t_o), \tag{8.41}$$

where the parameters u_{1_o} and t_f are computed as functions of t_o, x_{1_o} and x_{2_o} to insure that the trajectory described by (8.40) with initial conditions $x_1(t_o) = x_{1_o}$, $x_2(t_o) = x_{2_o}$ arrives at the desired terminal state: $x_1(t_f) = r_f$, $x_2(t_f) = 0$. The control u_2 is simply

$$u_2 = -1200\, x_3, \tag{8.42}$$

where the value of the coefficient was chosen in correspondence with the nature of the known optimal solutions.

TABLE 11. RESULTS OF SIMULATION OF THE CONTROLLED SYSTEM OF EXAMPLE C .

TRAJ.	INITIAL CONDITIONS			COST		COST PENALTY:
	$x_1(0)$ (ft)	$x_2(0)$ (ft/sec)	$x_3(0)$ (rad/sec)	Optimal	Using Synthesized Control	Percentage Above Optimal Cost
1	125000.	-229.80	0.0	52.94	53.77	1.57
2	125000.	-286.83	-0.1789×10^{-3}	94.31	96.59	2.42
3	125000.	-170.26	-0.1453×10^{-3}	30.60	31.90	4.25
4	125000.	-229.53	-0.1845×10^{-3}	54.72	54.95	0.42
5	125000.	-205.29	-0.3840×10^{-3}	49.04	49.70	1.35
6	125000.	-256.47	-0.3602×10^{-3}	76.83	76.83	0.00
7	125000.	-244.68	-0.5510×10^{-3}	78.85	79.66	1.03
8	125000.	-216.25	-0.6088×10^{-3}	65.60	67.53	2.94
9	125000.	-274.12	-0.7615×10^{-3}	116.94	120.72	3.23

B. SUGGESTIONS FOR FUTURE RESEARCH

In the literature comparatively little attention has been devoted to the development of computational procedures for the practical solution of the general optimal control problem by methods other than dynamic programming. However, the diversity of control problems occurring in engineering technology suggests that no one synthesis procedure can be well suited for all applications. It is important, therefore, that alternative general synthesis techniques be developed so that a variety of tools is available for the treatment of these problems. The method described in these pages represents one approach, and it suggests others. For example, approximation of the adjoint vector $\underline{\lambda}$ as a function of $\underline{x}$, or the control law $\underline{c}^*(\underline{x})$ itself could provide the basis for near-optimal control. Approximation by functions other than polynomials may also be advantageous. Any proposed synthesis procedure should be tested and evaluated with respect to computational feasibility and ease of implementation.

In Chapter III certain assumptions were made about the nature of the optimal cost function in order to provide a theoretical basis for the development of the synthesis procedure, and it was noted that direct verification of the validity of these assumptions is generally not possible. An interesting question worthy of exploration is the following: Under what more fundamental (and presumably more readily verifiable) or less restrictive assumptions on the nature of the problem than those made here can the relation (3.5) be established? Although some answers to this question are available [Ref. 16], certain assumptions must still be made which cannot be easily validated.

APPENDIX B. FLOW DIAGRAM OF THE APPROXIMATION PROGRAM

Figure 8 is a general flow diagram of the program to approximate $I(\underline{x})$ described in Chapter IV and used in the control law synthesis for the examples of Chapter VIII.

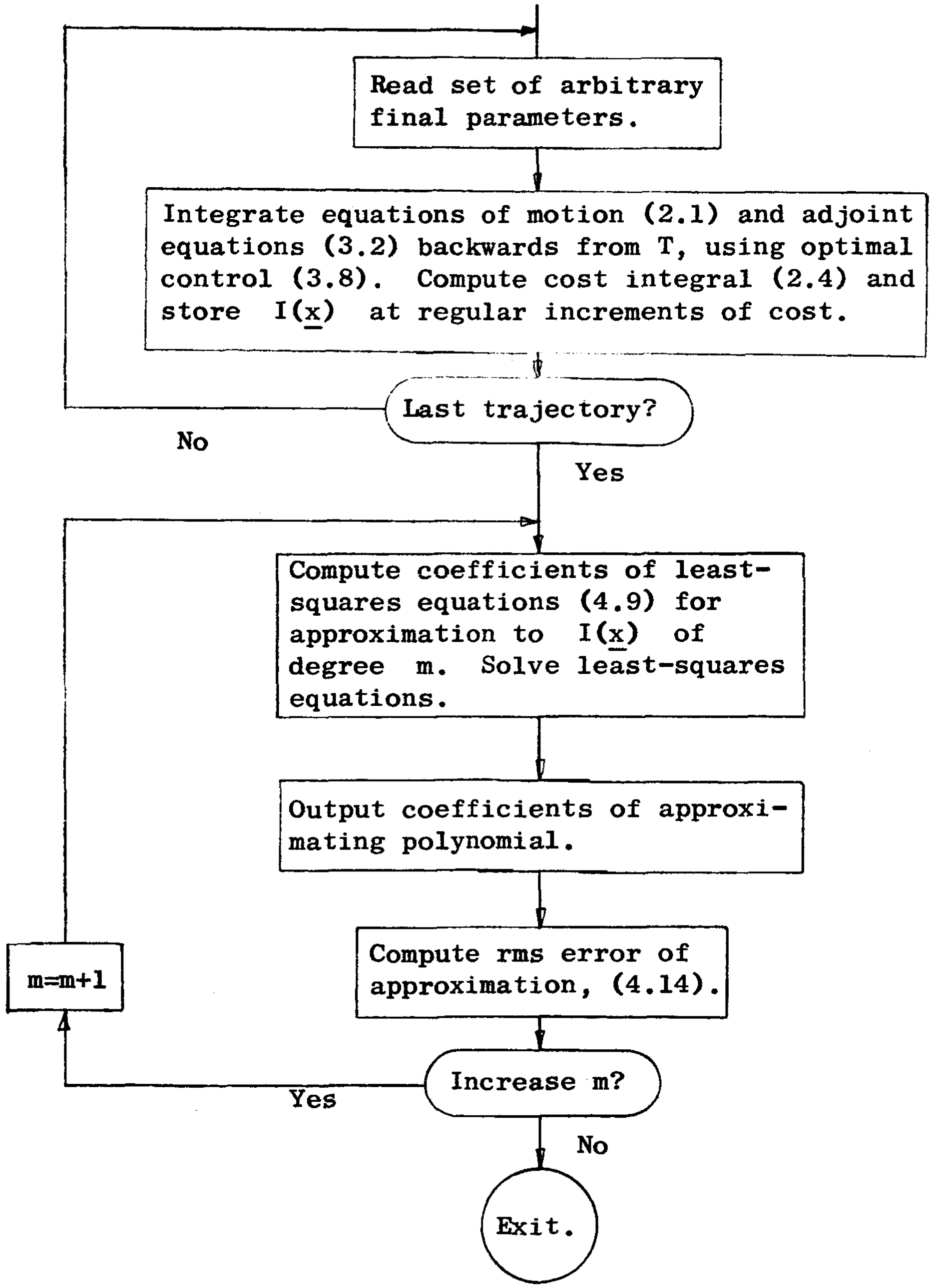


FIG. 8. FLOW DIAGRAM OF PROGRAM FOR APPROXIMATING $I(\underline{x})$.

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